Consistency and strong consistency of non-parametric estimate of the potential function of stationary and isotropic pairwise interaction point processes

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Abstract

The paper deals with the Gibbs point processes, which are usually used to model interaction between particles. These processes are defined and characterized through the Papangelou conditional intensity. In the paper, we assume that the model is a stationary pairwise interaction point process, so that its Papangelou conditional intensity involves two terms: a Poisson intensity parameter and a potential function. We suggest a new non-parametric estimator for the potential function in the Papangelou conditional intensity. Consistency and strong consistency for the resulting estimator of the total Papangelou conditional intensity are proved in the case of the finite-range interaction potential.

keywords: Kernel-type estimator, Pairwise interaction point process, Rates of strong uniform consistency, Papangelou conditional intensity, Spatial statistics.

1 Introduction

Gibbs point processes are a natural class of models for point patterns exhibiting interactions between the points. Fields of applications for point processes are image processing, analysis of the structure of tissues in medical sciences, forestry (see [15]), ecology (see [5]), spatial epidemiology (see [14]) and astrophysics (see [18]). Non-parametric estimation has been largely ignored by researchers.

One exception is the suggestion to use the non-parametric estimation of the pair correlation function and its approximate relation to the pair potential through the Percus-Yevic equation (see [6]). The approximation is a result of a cluster expansion method, and it is accurate only for sparse data. Pairwise interaction point processes densities are intractable as the normalizing constant is unknown and/or extremely complicated to approximate. However, we can resort to estimates of parameters using the conditional intensity. In this present paper, we assume that the pairwise interaction point process is stationary and isotropic , so that its Papangelou conditional intensity involves two terms: a Poisson intensity parameter and a pair potential function (or pairwise interaction function). This paper follows a previous one, that was interested in the estimator of the Poisson parameter in the Papangelou conditional intensity. In this present paper, we propose a new non-parametric estimation of the potential function in the Papangelou conditional intensity. We establish consistency and strong consistency for the resulting estimator.

Our paper is organized as follows. Section 2 introduces basic definitions and notations. In Section 3, we briefly present some models satisfying the assumptions needed to prove our asymptotic results. In Section 4, we present our main results. Consistency of the non-parametric estimator for the potential function in the Papangelou conditional intensity is proved in Section 4.1, it is based on the knowledge of the Papangelou conditional intensity and the iterated Georgii-Nguyen-Zessin formula. Using Orlicz spaces we can obtain a strong consistency of the non-parametric estimator in Section 4.2. The proofs are given in Section 5.

2 Basic notations

Throughout the paper we adopt the following notations. Let $\mathscr{B}^d$ be the Borel σ-algebra (generated by open sets) in $\mathbb{R}^d$ (the d-dimensional space) and $\mathscr{B}^d_O \subseteq \mathscr{B}^d$ denote the class of bounded Borel sets. We define a spatial point process $\mathbf{X}$ on $\mathbb{R}^d$ as a locally finite random subset of $\mathbb{R}^d$, i.e. the number of points $N(W) = n(\mathbf{X}_W)$ of the restriction of $\mathbf{X}$ to W is a finite random variable whenever $W \subset \mathbb{R}^d$ is a bounded region. $N_{lf} = \{\mathbf{x} \subseteq \mathbb{R}^d; n(\mathbf{x}_W) = n(\mathbf{x} \cap W) < \infty, \forall\, W \in \mathscr{B}^d_0\}$ is the space of locally finite configurations of points in $\mathbb{R}^d$ and will be denoted by $\mathbf{x}$. We equip N_{lf} with σ-algebra $\mathscr{N}_{lf} = \sigma\{\{\mathbf{x} \in N_{lf} : n(\mathbf{x}_W) = m\}, m \in \mathbb{N}_0, W \in \mathscr{B}^d_0\}$. That is $\mathscr{N}_{lf}$ is the smallest sigma algebra generated by $\{\mathbf{x} \in N_{lf} : n(\mathbf{x}_W) = m\}$. The volume of a bounded Borel set W of $\mathbb{R}^d$ is denoted by $|W|$ and o denotes the origin of $\mathbb{R}^d$, i.e. $o = (0, \dots, 0) \in \mathbb{R}^d$. For any finite subset Γ of $\mathbb{Z}^d$, we denote $|\Gamma|$ the number of elements in Γ. $||\cdot||$ denotes Euclidean distance on $\mathbb{R}^d$. $\sigma_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the measure of the unit ball in $\mathbb{R}^d$. Further, $\sum^{\neq}_{\xi_1,\dots,\xi_n}$ means that the summation goes over the n-tuples of mutually distinct points $\xi_1, \ldots, \xi_n$. Let $\mathbb{S}^{d-1}$ be the unit ball in $\mathbb{R}^d$.

For any Gibbs point processes in a bounded window, the Papangelou conditional intensity at a location u given the configuration $\mathbf{x}$ is related to the probability density f by $\lambda(u, \mathbf{x}) = f(\mathbf{x} \cup \{u\} / f(\mathbf{x})$ (for $u \notin \mathbf{x}$), the ratio of the probability densities for the configuration $\mathbf{x}$ with and without the point u added. The Papangelou conditional intensity can be interpreted as follows: for any $u \in \mathbb{R}^d$ and $\mathbf{x} \in N_{lf}$, $\lambda(u, \mathbf{x})\mathrm{d}u$ corresponds to the conditional probability of observing a point in a ball of volume $\mathrm{d}u$ around u given the rest of the point process is $\mathbf{x}$.

Gibbs point processes in $\mathbb{R}^d$ can be defined and characterized through the Papangelou conditional intensity (see [16]) which is a function $\lambda : \mathbb{R}^d \times N_{lf} \to \mathbb{R}_+$. The Georgii-Nguyen-Zessin (GNZ) formula (see [21], [24], [8], [20]) states that for any non-negative measurable function h on $\mathbb{R}^d \times N_{lf}$

$$\mathrm{E} \sum_{u \in \mathbf{X}} h(u, \mathbf{X} \setminus u) = \mathrm{E} \int_{\mathbb{R}^d} h(u, \mathbf{X}) \lambda(u, \mathbf{X}) \mathrm{d}u. \tag{1}$$

Using induction we obtain the iterated GNZ-formula: for non-negative functions $h : (\mathbb{R}^d)^n \times N_{lf} \longrightarrow \mathbb{R}$

$$\mathrm{E} \sum_{u_1, \ldots, u_n \in \mathbf{X}}^{\neq} h(u_1, \ldots, u_n, \mathbf{X} \setminus \{u_1, \ldots, u_n\}) \\ = \int \ldots \int \mathrm{E}\, h(u_1, \ldots, u_n, \mathbf{X}) \lambda(u_1, \ldots, u_n, \mathbf{X}) \mathrm{d}u_1 \ldots \mathrm{d}u_n \tag{2}$$

where $\lambda(u_1, \ldots, u_n, \mathbf{x})$ is Papangelou conditional intensity and is defined (not uniquely) by $\lambda(u_1, \ldots, u_n, \mathbf{x}) = \lambda(u_1, \mathbf{x}) \lambda(u_2, \mathbf{x} \cup \{u_1\}) \ldots \lambda(u_n, \mathbf{x} \cup \{u_1, \ldots, u_{n-1}\})$.

3 Examples of Papangelou conditional intensity

Examples of Papangelou conditional intensities are presented in [1], [16], [17]. The following presents some examples. Let $u \in \mathbb{R}^d$, $\mathbf{x} \in N_{lf}$ and $R > 0$.

1. A special case of pairwise interaction is the Strauss process. It has Papangelou conditional intensity

   $$\lambda(u, \mathbf{x}) = \beta \Phi^{n_{[0,R]}(u, \mathbf{x} \setminus u)}$$

   where $\beta > 0$, $0 \leq \Phi \leq 1$ and $n_{[0,R]}(u, \mathbf{x}) = \sum_{v \in \mathbf{x}} 1\!\!1(\|v - u\| \leq R)$ is the number of pairs in $\mathbf{x}$ with distance not greater than R.

2. Piecewise Strauss point process.

$$\lambda(u,\mathbf{x}) = \beta \prod_{j=1}^{p} \Phi_j^{n_{[R_{j-1},R_j]}(u,\mathbf{x}\backslash u)}$$

where $\beta > 0$, $0 \leq \Phi_j \leq 1$, $n_{[R_{j-1},R_j]}(u,\mathbf{x}) = \sum_{v\in\mathbf{x}} 1\!\!1(\|v-u\| \in [R_{j-1},R_j])$ and $R_0 = 0 < R_1 < \ldots < R_p = R < \infty$.

3. Triplets point process.

$$\lambda(u,\mathbf{x}) = \beta \Phi^{s_{[0,R]}(\mathbf{x}\cup u) - s_{[0,R]}(\mathbf{x}\backslash u)}$$

where $\beta > 0$, $0 \leq \Phi \leq 1$ and $s_{[0,R]}(\mathbf{x})$ is the number of unordered triplets that are closer than R.

4. Lennard-Jones model.

$$\lambda(u,\mathbf{x}) = \beta \prod_{v\in\mathbf{x}\backslash u} \Phi(\|v-u\|)$$

with $\log\Phi(r) = \left(\theta^6 r^{-6} - \theta^{12} r^{-12}\right) 1\!\!1_{(0,R]}(r)$, for $r = ||v-u||$, where $\theta > 0$ and $\beta > 0$ are parameters.

4 Main results

The Papangelou conditional intensity (see [16]) for a pairwise interaction point process is defined by

$$\lambda(u,\mathbf{x}) = \gamma_0(u)\exp\left(-\sum_{v\in\mathbf{x}\backslash u} \gamma_0(\{u,v\})\right).$$

If $\gamma_0(u) = \beta$ is a constant and $\gamma_0(\{u,v\}) = \gamma(||u-v||)$ is invariant under translations and rotations, then a pairwise interaction point process is said to be stationary and isotropic or homogeneous.

For convenience, throughout in this paper, we consider a stationary and isotropic pairwise interaction point process. Then its Papangelou conditional intensity at a location u is given by

$$\lambda(u,\mathbf{x}) = \beta^{\star}\exp\left(-\sum_{v\in\mathbf{x}\backslash u} \gamma(||v-u||)\right),\ \forall u\in\mathbb{R}^d,\ \mathbf{x}\in N_{lf} \tag{3}$$

where $\beta^\star$ is the so-called Poisson intensity parameter and γ is the so-called the pair potential and is assumed a non-negative function; a name that originates in statistical physics: it measures the potential energy caused by the interaction among pairs of points (u,v) as a function of their distance $||v-u||$. The pairwise interaction between points may also be described in terms of the pair potential function γ into the interaction function $\Phi = \exp(-\gamma)$ which has the following interpretation. For $\Phi > 1$, $\lambda(u,\mathbf{x})$ is increasing in $\mathbf{x}$ (the attractive case). For $\Phi < 1$, $\lambda(u,\mathbf{x})$ is decreasing in $\mathbf{x}$ (the repulsive case). It can be computed for the case $\Phi = 1$ which corresponds to the homogeneous Poisson point process with intensity $\beta^\star$. Usually a finite range of interaction R, is assumed such that

$$\gamma(||v-u||) = 0 \quad \text{whenever } ||v-u|| > R.$$

In other words, $\lambda(u,\mathbf{x})$ depends on $\mathbf{x}$ only through $\mathbf{x}\cap B(u,R)$, i.e.

$$\lambda(u,\mathbf{x}) = \lambda(u,\mathbf{x}\cap B(u,R)), \tag{4}$$

where $B(u,R)$ is the closed ball in $\mathbb{R}^d$ with centered at u and radius $R > 0$. In this present paper, we propose a new non-parametric estimation of the pair potential γ (or more precisely for the pairwise interaction function $\Phi = \exp(-\gamma)$) in the Papangelou conditional intensity for a stationary and isotropic pairwise interaction point process. We establish the consistency and the strong consistency for the resulting estimator.

Suppose that a single realization $\mathbf{x}$ of a point process $\mathbf{X}$ is observed in a bounded window $W_n \in \mathscr{B}_0^d$ where $(W_n)_{n\geq 1}$ is a sequence of cubes growing up to $\mathbb{R}^d$. Throughout in this paper, $\widetilde{h}$ is a non-negative measurable function defined for all $u \in \mathbb{R}^d$, $\mathbf{x} \in N_{lf}$ by

$$\widetilde{h}(u,\mathbf{x}) = 1\!\!1\left(\inf_{v\in\mathbf{x}} ||v-u|| > R\right) = 1\!\!1\left(d(u,\mathbf{x}) > R\right), \tag{5}$$

note that for $r \in (0,R]$,

$$\widetilde{F}(o,rv) = \mathrm{E}[\widetilde{h}(o,\mathbf{X})\widetilde{h}(rv,\mathbf{X})] = \mathbb{P}(d(o,\mathbf{X}) > R, d(rv,\mathbf{X}) > R)$$

and

$$J(r) = \int_{\mathbb{S}^{d-1}} \widetilde{F}(o,rv)\mathrm{d}v.$$

To estimate the function $\beta^{\star 2}J(r)\Phi(r)$, we suggest an edge-corrected kernel-type estimator $\widehat{R}_n(r)$ defined for $r \in (0,R]$ by

$$\widehat{R}_n(r) = \frac{1}{b_n|W_{n\ominus 2R}|\sigma_d} \sum_{\substack{u,v\in\mathbf{X} \\ ||v-u||\leq R}}^{\neq} \frac{1\!\!1_{W_{n\ominus 2R}}(u)}{||v-u||^{d-1}} \widetilde{h}(u,\mathbf{X}\setminus\{u,v\})\widetilde{h}(v,\mathbf{X}\setminus\{u,v\})K_1\left(\frac{||v-u||-r}{b_n}\right). \tag{6}$$

$\ominus$ will denote Minkowski substraction, with the convention that

$$W_{n\ominus 2R} = W_n \ominus B(u,2R) = \{u \in W_n : ||u-v|| \leq 2R \quad \text{for all} \quad v \in W_n\}$$

denotes the $2R$-interior of the cubes W_n, with Lebesgue measure $|W_{n\ominus 2R}| > 0$. K_1 is an univariate kernel function associated with a positive sequence $(b_n)_{n\geq 1}$ of bandwidths satisfying the following:

Condition $K(1,\alpha)$: The sequence of bandwidths $b_n > 0$ for $n \geq 1$, is chosen such that

$$\lim_{n\to\infty} b_n = 0 \quad \text{and} \quad \lim_{n\to\infty} b_n |W_{n\ominus 2R}| = \infty.$$

The kernel function $K_1 : \mathbb{R} \longrightarrow \mathbb{R}$ is non-negative and bounded with bounded support, such that:

$$\int_{\mathbb{R}} K_1(\rho)\mathrm{d}\rho = 1, \int_{\mathbb{R}} \rho^j K_1(\rho)\mathrm{d}\rho = 0, j = 0,1,\ldots,\alpha-1, \quad \text{for } \alpha \geq 2.$$

To estimate the function $\beta^\star J(r)$, we suggest an empirical estimator $\widehat{J_n}(r)$ defined for $r \in (0,R]$ by

$$\widehat{J_n}(r) = \frac{1}{|W_{n\ominus 2R}|} \sum_{u\in\mathbf{X}} 1\!\!1_{W_{n\ominus 2R}}(u)\widetilde{h}(u,\mathbf{X}\setminus\{u\})h^\star(u,\mathbf{X}\setminus\{u\}), \tag{7}$$

where $h^\star(u,\mathbf{x}) = \int_{\mathbb{S}^{d-1}} \widetilde{h}(rv+u,\mathbf{x})\mathrm{d}v$. Using the spatial ergodic theorem of [19], the estimator (7) turns out to be unbiased and strongly consistent. The natural estimator of the Poisson intensity $\beta^\star$ is

$$\widehat{\beta}_n = \frac{\sum_{u\in\mathbf{X}} 1\!\!1_{\Lambda_{n,R}}(u)\widetilde{h}(u,\mathbf{X}\setminus\{u\})}{\int_{\Lambda_{n,R}} \widetilde{h}(u,\mathbf{X})\mathrm{d}u}. \tag{8}$$

In [3], we introduced a semi-parametric estimator (8) of the parameter $\beta^\star$ and studied its strong consistency, asymptotic normality and simulation study. Finally, the desired estimator for the interaction function $\Phi(r) = \exp(-\gamma(r))$ is defined by the ratio

$$\widehat{\Phi}_n(r) = \frac{\widehat{R}_n(r)}{\widehat{\beta}_n\widehat{J}_n(r)}, \quad \text{for} \quad r \in (0,R]. \tag{9}$$

The strong consistency of the estimators (7) and (8) implies the following:

Proposition 1. *Let γ be a pairwise interaction potential defined in* (3) *satisfying condition* (4)*. Let K_1 kernel function satisfying Condition $K(1,\alpha)$ and the function*

$J(r)exp(-\gamma(r))$ *has bounded and continuous partial derivatives of order* α *for all* $\alpha \geq 1$ *in* $(r-\delta, r+\delta)$ *for some* $\delta > 0$*. Then as* $n \to \infty$

$$\widehat{\Phi}_n(r) \longrightarrow \exp(-\gamma(r)) \quad \textit{in probability} \quad \mathbb{P} \quad (\textit{resp.}\mathbb{P}\textit{-a.s.}) \quad \textit{iff}$$

$$\widehat{R}_n(r) \longrightarrow \beta^{\star^2} J(r) \exp(-\gamma(r)) \quad \textit{in probability} \quad \mathbb{P} \quad (\textit{resp.}\mathbb{P}\textit{- a.s.}).$$

The convergence in probability (consistency) of the kernel-type estimator $\widehat{R}_n(r)$ (defined in (6)) will be discussed in Section 4.1. Conditions ensuring uniform $\mathbb{P}$-a.s. convergence (strong uniform consistency) of the kernel-type estimator $\widehat{R}_n(r)$ will be discussed in Section 4.2.

4.1 Consistency of the kernel-type estimator

In this section we discuss the consistency of $\widehat{R}_n(r)$. For this it follows the interesting to determine the asymptotic behavior of $\mathrm{E}\widehat{R}_n(r)$ and $\mathrm{Var}\widehat{R}_n(r)$.

4.1.1 Asymptotic representation of the mean and the variance of the kernel-type estimator

In this section we derive asymptotic representations for the mean and the variance of the kernel-type estimator $\widehat{R}_n(r)$. We will use the Landau notation $f(n) = \mathscr{O}(h(n))$ as $n \to \infty$ for error terms $f(n)$ satisfying $\limsup_{n\to\infty} \frac{f(n)}{h(n)} < \infty$.

Theorem 1. *Let* γ *be a pairwise interaction potential defined in* (3) *satisfying condition* (4)*. Let* K_1 *kernel function satisfying Condition* $K(1,1)$*. Then we have*

$$\lim_{n\to\infty} \mathrm{E}\widehat{R}_n(r) = \beta^{\star 2} J(r) \exp(-\gamma(r)),$$

in every point of continuity $r \in (0,R]$ *of* $J \times \exp(-\gamma)$.

If Condition $K(1,\alpha)$ *is satisfied and the function* $\exp(-\gamma(r))J(r)$ *has bounded and continuous partial derivatives of order* α *in* $(r-\delta, r+\delta)$ *for some* $\delta > 0$ *and for* $\alpha \geq 1$*. Then we have*

$$\mathrm{E}\widehat{R}_n(r) = \beta^{\star 2} J(r) \exp(-\gamma(r)) + \mathscr{O}(b_n^{\alpha}) \quad \textit{as} \ \ n \to \infty.$$

Theorem 2. *Let* γ *be a pairwise interaction potential defined in* (3) *satisfying condition* (4)*. Let* K_1 *kernel function satisfying Condition* $K(1,\alpha)$ *for all* $\alpha \geq 1$ *such that* $\int_{\mathbb{R}} K_1^2(\rho) \mathrm{d}\rho < \infty$ *. Then, we have,*

$$\lim_{n\to\infty} b_n |W_{n\ominus 2R}| \, \mathrm{Var}(\widehat{R}_n(r)) = \frac{2\beta^{\star 2}}{\sigma_d r^{d-1}} J(r) \exp(-\gamma(r)) \int_{\mathbb{R}} K_1^2(\rho) \mathrm{d}\rho,$$

in every point of continuity $r \in (0,R]$ *of* $J \times \exp(-\gamma)$.

4.2 Rates of strong uniform consistency of the kernel-type estimator

Before realizing the strong consistency for $\widehat{R}_n(r)$ (defined in (6)) we introduce some necessary definitions and notations. A Young function ψ is a real convex nondecreasing function defined on $\mathbb{R}^+$ which satisfies $\lim_{t\to\infty}\psi(t)=+\infty$ and $\psi(0)=0$. We define the Orlicz space L_ψ as the space of real random variables Z defined on the probability space $(\Omega,\mathscr{F},\mathbb{P})$ such that $E[\psi(|Z|/c)]<+\infty$ for some $c>0$. The Orlicz space L_ψ equipped with the so-called Luxemburg norm $\|.\|_\psi$ defined for any real random variable Z by

$$\|Z\|_\psi=\inf\{\,c>0\,;\,E[\psi(|Z|/c)]\le 1\,\}$$

is a Banach space. For more about Young functions and Orlicz spaces one can refer to [13]. Let $\theta>0$. We denote by ψ_θ the Young function defined for any $x\in\mathbb{R}^+$ by

$$\psi_\theta(x)=\exp((x+\xi_\theta)^\theta)-\exp(\xi_\theta^\theta)\quad\text{where}\quad \xi_\theta=((1-\theta)/\theta)^{1/\theta}\,\mathbb{1}\{0<\theta<1\}.$$

On the lattice $\mathbb{Z}^d$ we define the lexicographic order as follows: if $i=(i_1,\dots,i_d)$ and $j=(j_1,\dots,j_d)$ are distinct elements of $\mathbb{Z}^d$, the notation $i<_{lex}j$ means that either $i_1<j_1$ or for some p in $\{2,3,\dots,d\}$, $i_p<j_p$ and $i_q=j_q$ for $1\le q<p$. Let the sets $\{V_i^k\,;\,i\in\mathbb{Z}^d\,,\,k\in\mathbb{N}^*\}$ be defined as follows:

$$V_i^1=\{j\in\mathbb{Z}^d\,;\,j<_{lex}i\},$$

and for $k\ge 2$

$$V_i^k=V_i^1\cap\{j\in\mathbb{Z}^d\,;\,|i-j|\ge k\}\quad\text{where}\quad |i-j|=\max_{1\le l\le d}|i_l-j_l|.$$

By a real random field we mean any family $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ of real-valued random variables and for any subset Γ of $\mathbb{Z}^d$ define $\mathscr{F}_\Gamma=\sigma(\varepsilon_i\,;\,i\in\Gamma)$ and set

$$E_{|k|}(\varepsilon_i)=E(\varepsilon_i|\mathscr{F}_{V_i^{|k|}}),\quad k\in V_i^1.$$

Denote $\theta(q)=2q/(2-q)$ for $0<q<2$.

Next we list a set of conditions which are needed to obtain (rates of) strong uniform consistency over some compact set $[r_1,r_2]$ in $(0,R]$ of the estimator $\widehat{R}_n(r)$ to the function $\beta^{\star 2}J(r)\Phi(r)$. The following assumption is imposed:

Condition $\mathscr{L}p$: The kernel function K satisfies a Lipschitz condition, i.e. there exists a constant $\eta>0$ such that

$$\left|K_1(\rho)-K_1(\rho')\right|\le\eta|\rho-\rho'|\quad\text{for all}\quad \rho,\rho'\in[r_1,r_2].$$

Now, we split up the window $W_{n\ominus 2R}$ into cubes such as $W_{n\ominus 2R}=\cup_{i\in\Gamma_n}\Lambda_i$, following [22], [12], [10] we will describe a point process in $\mathbb{R}^d$ as lattice process by means of this decomposition $\Lambda_i=\{\xi\in\mathbb{R}^d;\widetilde{q}(i_j-\frac{1}{2})\leq\xi_j\leq\widetilde{q}(i_j+\frac{1}{2}), j=1,\dots,d\}$ for a fixed number $\widetilde{q}>0$, $i=(i_1,\dots,i_d)$, where the process is observed in $W_{n\ominus 2R}=\cup_{i\in\widetilde{\Gamma}_n}\Lambda_i$, where $\widetilde{\Gamma}_n=\{i\in\Gamma_n; |i-j|\leq 1, \text{ for all } j\in\Gamma_n\}$, and the norm is $|j|=\max\{|j_1|,\dots,|j_d|\}$ and assume that Γ_n increases towards $\mathbb{Z}^d$ and we split up $\widehat{R}_n(r)$ as follows:

$$\widehat{R}_n(r)=\frac{1}{b_n|W_{n\ominus 2R}|\sigma_d}\sum_{i\in\Gamma_n}R_i(r),$$

where

$$R_i(r)=\sum_{\substack{u,v\in\mathbf{X}\\ ||v-u||\leq R}}^{\neq}\frac{1\!\!1_{\Lambda_i}(u)}{||v-u||^{d-1}}\widetilde{h}(u,\mathbf{X}\setminus\{u,v\})\widetilde{h}(v,\mathbf{X}\setminus\{u,v\})K_1\left(\frac{||v-u||-r}{b_n}\right).$$

Note for all $k\in\Gamma_n$, $\bar{R}_k=R_k(r)-\mathrm{E}\,R_k(r)$ and $S_n=\sum_{k\in\Gamma_n}\bar{R}_k(r)$.

Strong uniform consistency for $\widehat{R}_n(r)$ is obtained via assumptions of belonging to Orlicz spaces induced by exponential Young functions for stationary real random fields which allows us to derive the Kahane-Khintchine inequalities by [7].

Proposition 2. *We assume that Conditions $K(1,\alpha)$ and $\mathscr{L}p$ are fulfilled. Furthermore, we also assume that $J(r)\exp(-\gamma(r))$ has bounded and continuous partial derivatives of order α in $[r_1-\delta,r_2+\delta]$ for some $\delta>0$.*

If there exists $0<q<2$ such that $\bar{R}_0\in\mathbb{L}_{\psi_{\theta(q)}}$ and

$$\sum_{k\in V_0^1}\left\|\sqrt{\left|\bar{R}_kE_{|k|}(\bar{R}_0)\right|}\right\|^2_{\psi_{\theta(q)}}<\infty. \tag{10}$$

Then

$$\sup_{r_1\leq r\leq r_2}\left|\widehat{R}_n(r)-\beta^{\star 2}J(r)\exp(-\gamma(r))\right|=\mathscr{O}_{a.s.}\left(\frac{(\log n)^{1/q}}{b_n n^{d/2}}\right)+\mathscr{O}(b_n^{\alpha})\quad \textit{as}\quad n\to\infty.$$

Our results also carry through the most important particular case of Orlicz spaces random fields, for p-integrable $(2<p<+\infty)$ real random fields.

Proposition 3. *We assume that Conditions $K(1,\alpha)$ and $\mathscr{L}p$ are fulfilled. Furthermore, we also assume that $J(r)\exp(-\gamma(r))$ has bounded and continuous partial derivatives of order α in $[r_1-\delta,r_2+\delta]$ for some $\delta>0$.*

If there exists $p > 2$ such that $\bar{R}_0 \in \mathbb{L}^p$ and

$$\sum_{k \in V_0^1} \left\| \bar{R}_k E_{|k|}(\bar{R}_0) \right\|_{\frac{p}{2}} < \infty. \tag{11}$$

Assume that $b_n = n^{-q_2}(\log n)^{q_1}$ for some $q_1, q_2 > 0$. Let $a, b \geq 0$ be fixed and if $a(p+1) - d^2/2 - q_2 > 1 \quad$ and $\quad b(p+1) + q_1 > 1$. Then

$$\sup_{r_1 \leq r \leq r_2} \left| \widehat{R}_n(r) - \beta^{\star 2} J(r) \exp(-\gamma(r)) \right| = \mathscr{O}_{a.s.}\left(\frac{n^a (\log n)^b}{b_n n^{d/2}} \right) + \mathscr{O}(b_n^{\alpha}) \quad as \quad n \to \infty.$$

5 Proofs

Proof of Theorem 1. Using the notation

$$\widetilde{L}(u_1, \ldots, u_s, \mathbf{X}) = \widetilde{h}(u_1, \mathbf{X}) \ldots \widetilde{h}(u_s, \mathbf{X}), \tag{12}$$

where $\widetilde{h}$ is given by (5).

$$\widetilde{F}(u_1, \ldots, u_s) = \mathrm{E}[\widetilde{h}(u_1, \mathbf{X}) \ldots \widetilde{h}(u_s, \mathbf{X})],$$
$$\widetilde{J}(||u_1||, \ldots, ||u_s||) = 1\!\!1(||u_1|| \leq R, \ldots, ||u_s|| \leq R).$$

The calculation of the expectation and the variance of $\widehat{R}_n(r)$ is based on the iterated Georgii-Nguyen-Zessin (GNZ) formula (2), i.e. applying the preceding formula (2) for $s = 2$ and with

$$h(u, v, \mathbf{X}) = \frac{1\!\!1_{W_{n \ominus 2R}}(u)}{||v-u||^{d-1}} \widetilde{J}(||v-u||) \widetilde{L}(u, v, \mathbf{X}) K_1\left(\frac{||v-u|| - r}{b_n} \right),$$

we derive

$$\mathrm{E}\widehat{R}_n(r) = \frac{1}{b_n |W_{n \ominus 2R}| \sigma_d}$$
$$\mathrm{E} \int_{\mathbb{R}^{2d}} \frac{1\!\!1_{W_{n \ominus 2R}}(u)}{||v-u||^{d-1}} \widetilde{J}(||v-u||) \widetilde{L}(u, v, \mathbf{X}) K_1\left(\frac{||v-u|| - r}{b_n} \right) \lambda(u, v, \mathbf{X}) \mathrm{d}u \mathrm{d}v.$$

We remember the second order Papangelou conditional intensity by:

$$\lambda(u, v, \mathbf{x}) = \lambda(u, \mathbf{x}) \lambda(v, \mathbf{x} \cup \{u\}) \quad \text{for any } u, v \in \mathbb{R}^d \text{ and } \mathbf{x} \in N_{lf}.$$

Using the finite range property (4) for each function $\lambda(u, \mathbf{x})$ and $\lambda(v, \mathbf{x} \cup \{u\})$, we have

$$\begin{aligned} \lambda(u, \mathbf{X}) &= \lambda(u, \mathbf{X} \cap B(u, R)) \\ &= \beta^{\star} \quad \text{when} \quad d(u, \mathbf{X}) > R \end{aligned}$$

and

$$\lambda(v,\mathbf{X}\cup\{u\}) = \lambda(v,(\mathbf{X}\cap B(v,R))\cup\{u\}) \\ = \beta^\star \Phi(||v-u||) \quad \text{when} \quad d(v,\mathbf{X}) > R.$$

Consequently, by the stationarity of $\mathbf{X}$ and from the definition of $\widetilde{L}$ given by (5), we get

$$\begin{aligned}
\mathrm{E}\widehat{R}_n(r) &= \frac{\beta^{\star 2}}{b_n|W_{n\ominus 2R}|\sigma_d} \\
&\quad \mathrm{E}\int_{\mathbb{R}^{2d}} \frac{1\!\!1_{W_{n\ominus 2R}}(u)}{||v-u||^{d-1}} \widetilde{J}(||v-u||)\widetilde{L}(u,v,\mathbf{X})K_1\left(\frac{||v-u||-r}{b_n}\right)\Phi(||v-u||)\mathrm{d}u\mathrm{d}v \\
&= \frac{\beta^{\star 2}}{b_n\sigma_d}\int_{\mathbb{R}^d} \frac{\widetilde{J}(||s||)}{||s||^{d-1}}\mathrm{E}[\widetilde{L}(o,s,\mathbf{X})]K_1\left(\frac{||s||-r}{b_n}\right)\Phi(||s||)\mathrm{d}s.
\end{aligned}$$

Recall a property of the integration theory (see [2] or [23]). Let $\mathbb{S}^{d-1}$ be the unit ball in $\mathbb{R}^d$, i.e. $\mathbb{S}^{d-1} = \{u \in \mathbb{R}^d : ||u|| = 1\}$, then for any Borel function $f : \mathbb{R}^d \longrightarrow \mathbb{R}_+$,

$$\int_{\mathbb{R}^d} f(||u||)\mathrm{d}u = \int_0^\infty \int_{\mathbb{S}^{d-1}} f(rz)r^{d-1}\sigma_d \mathrm{d}r\mathrm{d}z.$$

By combining the above result, we get so:

$$\mathrm{E}\widehat{R}_n(r) = \frac{\beta^{\star 2}}{\sigma_d}\int_{-r/b_n}^\infty \int_{\mathbb{S}^{d-1}} \widetilde{J}(b_n\rho+r)\widetilde{F}(o,(b_n\rho+r)v)K_1(\rho)\Phi(b_n\rho+r)\sigma_d\mathrm{d}\rho\mathrm{d}v.$$

With bounded support on the kernel function and by dominated convergence theorem, we get as $n\to\infty$, $\mathrm{E}\widehat{R}_n(r) \longrightarrow \beta^{\star 2}J(r)\exp(-\gamma(r))$.

Now, we are going to prove the second part of Theorem 1. We have a product of two functions $\widetilde{F}(o,(b_n\rho+r)v)\Phi(b_n\rho+r)$ and we approximate each one of them with a Taylor formula up to a certain α. For any point ρ in $\mathbb{R}$, there exists $\theta \in (0,1)$, such that by Taylor-Lagrange formula, we get

$$\Phi(b_n\rho+r) = \Phi(r) + \sum_{k=1}^{\alpha-1}\frac{\Phi^{(k)}(r)}{k!}(b_n\rho)^k + \frac{\Phi^{(\alpha)}(r+b_n\rho\theta)}{\alpha!}b_n^\alpha$$

and

$$\widetilde{F}(o,(b_n\rho+r)v) = \widetilde{F}(o,rv) + \sum_{k=1}^{\alpha-1}\frac{\widetilde{F}^{(k)}(o,rv)}{k!}(b_n\rho)^k + \frac{\widetilde{F}^{(\alpha)}(o,(r+b_n\rho\theta)v)}{\alpha!}b_n^\alpha.$$

So we multiply two such functions, their product equals the product of their α^{th} Taylor polynomials plus terms involving powers of r higher than α. In other words, to compute the α^{th} Taylor polynomial of a product of two functions, find the product of their Taylor polynomials, ignoring powers of r higher than α. So we denote this product by $T_n(rv,r)$, then we have as $n \to \infty$

$$\widetilde{F}(o,(b_n\rho+r)v)\Phi(b_n\rho+r) = \widetilde{F}(o,rv)\Phi(r) + \sum_{k=1}^{\alpha-1} T_n(rv,r)(b_n\rho)^k + \mathcal{O}(b_n^{\alpha}).$$

It follows that,

$$\begin{aligned} \mathrm{E}\widehat{R}_n(r) &= \beta^{\star 2} J(r)\Phi(r) \\ &\quad + \beta^{\star 2} \int_{\mathbb{S}^{d-1}} \sum_{k=1}^{\alpha-1} T_n(rv,r) b_n^k \mathrm{d}v \int_{\mathbb{R}} \rho^k K_1(\rho)\mathrm{d}\rho \\ &\quad + \mathcal{O}(b_n^{\alpha}) \quad \text{as } n \to \infty. \end{aligned}$$

Together with Condition $K(1,\alpha)$ implies the second assertion of Theorem 1. □

Proof of Theorem 2. The proof of Theorem 2 makes use of the following corollary.

Corollary 1. *Consider any Gibbs point process* $\mathbf{X}$ *in* $\mathbb{R}^d$ *with Papangelou conditional intensity* λ. *For any non-negative, measurable and symmetric function* $f : \mathbb{R}^d \times \mathbb{R}^d \times N_{lf} \longrightarrow \mathbb{R}$, *we have*

$$\begin{aligned} &\mathrm{Var}\left(\sum_{u,v\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\setminus\{u,v\}) \right) \\ &= 2\,\mathrm{E}\int_{\mathbb{R}^{2d}} f^2(u,v,\mathbf{X})\lambda(u,v,\mathbf{X})\mathrm{d}u\mathrm{d}v \\ &+ 4\,\mathrm{E}\int_{\mathbb{R}^{3d}} f(u,v,\mathbf{X}) f(v,w,\mathbf{X})\lambda(u,v,w,\mathbf{X})\mathrm{d}u\mathrm{d}v\mathrm{d}w \\ &+ \mathrm{E}\int_{\mathbb{R}^{4d}} f(u,v,\mathbf{X}) f(w,y,\mathbf{X})\lambda(u,v,w,y,\mathbf{X})\mathrm{d}u\mathrm{d}v\mathrm{d}w\mathrm{d}y \\ &- \int_{\mathbb{R}^{4d}} \mathrm{E}[f(u,v,\mathbf{X})\lambda(u,v,\mathbf{X})]\,\mathrm{E}[f(w,y,\mathbf{X})\lambda(w,y,\mathbf{X})]\mathrm{d}u\mathrm{d}v\mathrm{d}w\mathrm{d}y. \end{aligned}$$

Proof. Consider the decomposition (see [11] and [9])

$$\Big(\sum_{u,v\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\backslash\{u,v\})\Big)^2 = 2\sum_{u,v\in\mathbf{X}}^{\neq} f^2(u,v,\mathbf{X}\backslash\{u,v\})$$
$$+4\sum_{u,v,w\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\backslash\{u,v,w\})f(v,w,\mathbf{X}\backslash\{u,v,w\})$$
$$+\sum_{u,v,w,y\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\backslash\{u,v,w,y\})f(w,y,\mathbf{X}\backslash\{u,v,w,y\}). \tag{13}$$

Applying the preceding (GNZ) formula (2) combining with (13), we obtain

$$\begin{aligned}
&\operatorname{Var}\sum_{u,v\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\backslash\{u,v\})\\
&=\mathrm{E}\big(\sum_{u,v\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\backslash\{u,v\})\big)^2-\big(\mathrm{E}\sum_{u,v\in\mathbf{X}}^{\neq} f(u,v,\mathbf{X}\backslash\{u,v\})\big)^2\\
&=2\,\mathrm{E}\int_{\mathbb{R}^{2d}} f^2(u,v,\mathbf{X})\lambda(u,v,\mathbf{X})\mathrm{d}u\mathrm{d}v\\
&+4\,\mathrm{E}\int_{\mathbb{R}^{3d}} f(u,v,\mathbf{X})f(v,w,\mathbf{X})\lambda(u,v,w,\mathbf{X})\mathrm{d}u\mathrm{d}v\mathrm{d}w\\
&+\mathrm{E}\int_{\mathbb{R}^{4d}} f(u,v,\mathbf{X})f(w,y,\mathbf{X})\lambda(u,v,w,y,\mathbf{X})\mathrm{d}u\mathrm{d}v\mathrm{d}w\mathrm{d}y\\
&-\int_{\mathbb{R}^{4d}} \mathrm{E}[f(u,v,\mathbf{X})\lambda(u,v,\mathbf{X})]\,\mathrm{E}[f(w,y,\mathbf{X})\lambda(w,y,\mathbf{X})]\mathrm{d}u\mathrm{d}v\mathrm{d}w\mathrm{d}y.
\end{aligned}$$

We obtain the desired result. □

Applying Corollary 1 to this function

$$f(u,v,\mathbf{X})=\frac{1\!\!1_{W_{n\ominus 2R}}(u)}{||v-u||^{d-1}}\widetilde{J}(||v-u||)\widetilde{L}(u,v,\mathbf{X})K_1\left(\frac{||v-u||-r}{b_n}\right),$$

it is easily seen that $\operatorname{Var}\widehat{R}_n(r) = A_1 + A_2 + A_3 - A_4$, where

$$A_1 = \frac{2}{b_n^2 |W_{n\ominus 2R}|^2 \sigma_d^2}$$
$$\mathrm{E}\int_{\mathbb{R}^{2d}} \frac{1\!\!1_{W_{n\ominus 2R}}(u)}{||v-u||^{2(d-1)}} \widetilde{J}(||v-u||)\widetilde{L}(u,v,\mathbf{X}) K_1^2\left(\frac{||v-u||-r}{b_n}\right)\lambda(u,v,\mathbf{X})\mathrm{d}u\mathrm{d}v,$$
$$A_2 = \frac{4}{b_n^2 |W_{n\ominus 2R}|^2 \sigma_d^2}\,\mathrm{E}\int_{\mathbb{R}^{3d}} \frac{1\!\!1_{W_{n\ominus 2R}}(u)1\!\!1_{W_{n\ominus 2R}}(v)}{||v-u||^{d-1}||v-w||^{d-1}} \widetilde{J}(||v-u||,||v-w||)\widetilde{L}(u,v,w,\mathbf{X})$$
$$K_1\left(\frac{||v-u||-r}{b_n}\right)K_1\left(\frac{||v-w||-r}{b_n}\right)\lambda(u,v,w,\mathbf{X})\mathrm{d}u\mathrm{d}v\mathrm{d}w,$$
$$A_3 = \frac{1}{b_n^2 |W_{n\ominus 2R}|^2 \sigma_d^2}\,\mathrm{E}\int_{\mathbb{R}^{4d}} \frac{1\!\!1_{W_{n\ominus 2R}}(u)1\!\!1_{W_{n\ominus 2R}}(w)}{||v-u||^{d-1}||w-y||^{d-1}} \widetilde{J}(||v-u||,||w-y||)\widetilde{L}(u,v,w,y,\mathbf{X})$$
$$\times K_1\left(\frac{||v-u||-r}{b_n}\right)K_1\left(\frac{||w-y||-r}{b_n}\right)\lambda(u,v,w,y,\mathbf{X})\mathrm{d}u\mathrm{d}v\mathrm{d}w\mathrm{d}y,$$

and

$$A_4 = \frac{1}{b_n^2 |W_{n\ominus 2R}|^2 \sigma_d^2}$$
$$\left(\mathrm{E}\int_{\mathbb{R}^{d}} \frac{1\!\!1_{W_{n\ominus 2R}}(u)}{||v-u||^{d-1}} \widetilde{J}(||v-u||)\widetilde{L}(u,v,\mathbf{X}) K_1\left(\frac{||v-u||-r}{b_n}\right)\lambda(u,v,\mathbf{X})\mathrm{d}u\mathrm{d}v\right)^2.$$

The asymptotic behavior of the leading term A_1 is obtained by applying the second order Papangelou conditional intensity given by:

$$\lambda(u,v,\mathbf{x}) = \lambda(u,\mathbf{x})\lambda(v,\mathbf{x}\cup\{u\}) \quad \text{for any } u,v\in\mathbb{R}^d \quad \text{and } \mathbf{x}\in N_{lf}.$$

Using the finite range property (4) for each function $\lambda(u,\mathbf{x})$ and $\lambda(v,\mathbf{x}\cup\{u\})$ and by stationarity of $\mathbf{X}$, it results

$$A_1 = \frac{2\beta^{\star 2}}{b_n^2 |W_{n\ominus 2R}|^2 \sigma_d^2}$$
$$\int_{\mathbb{R}^{2d}} \frac{1\!\!1_{W_{n\ominus 2R}}(u)}{||v-u||^{2(d-1)}} \widetilde{J}(||v-u||)\,\mathrm{E}[\widetilde{L}(o,v-u,\mathbf{X})] K_1^2\left(\frac{||v-u||-r}{b_n}\right)\Phi(||v-u||)\mathrm{d}u\mathrm{d}v$$
$$= \frac{2\beta^{\star 2}}{b_n |W_{n\ominus 2R}| \sigma_d^2}\int_{-r/b_n}^{\infty}\int_{\mathbb{S}^{d-1}} \frac{\widetilde{J}(b_n\rho+r)}{(b_n\rho+r)^{d-1}}\widetilde{F}(o,(b_n\rho+r)w)K_1^2(\rho)\Phi(b_n\rho+r)\mathrm{d}\rho\mathrm{d}\sigma(w).$$

Dominated convergence theorem and assumption of K_1 imply for $r\in(0,R]$

$$\lim_{n\to\infty} b_n|W_{n\ominus 2R}|A_1 = \frac{2\beta^{\star 2}}{\sigma_d r^{d-1}}J(r)\Phi(r)\int_{\mathbb{R}} K_1^2(\rho)\mathrm{d}\rho.$$

We will now show that all other integrals to $\mathrm{Var}(\widehat{R}_n(r))$ converge to zero. For the asymptotic behavior of the second term A_2, we remember the third order Papangelou conditional intensity by

$$\lambda(u,v,w,\mathbf{x}) = \lambda(u,\mathbf{x})\lambda(v,\mathbf{x}\cup\{u\})\lambda(w,\mathbf{x}\cup\{u,v\})$$

for any $u,v,w \in \mathbb{R}^d$ and $\mathbf{x} \in N_{lf}$. Since $\mathbf{X}$ is a point process to interact in pairs, the interaction terms due to triplets or higher order are equal to one, i.e. the potential $\gamma(\mathbf{y}) = 0$ when $n(\mathbf{y}) \geq 3$, for $\emptyset \neq \mathbf{y} \subseteq \mathbf{x}$. Using the finite range property (4) for each function $\lambda(u,\mathbf{x})$, $\lambda(v,\mathbf{x}\cup\{u\})$ and $\lambda(w,\mathbf{x}\cup\{u,v\})$ and after an elementary calculation, we have

$$\lambda(u,v,w,\emptyset) = \begin{cases} \beta^{\star 3}\Phi(||v-u||)\Phi(||w-v||) & \text{if} \quad ||u-w|| < R \\ \beta^{\star 3}\Phi(||v-u||) & \text{otherwise.} \end{cases}$$

Which ensures that $\lambda(u,v,w,\emptyset)$ is a function that depends only variables $||v-u||, ||w-v||$, denoted by $\Phi_1(||v-u||,||w-v||)$.

According to the stationarity of $\mathbf{X}$, it follows that

$$\begin{aligned}
A_2 &= \frac{4}{b_n^2 |W_{n\ominus 2R}|^2 \sigma_d^2} \mathrm{E} \int_{\mathbb{R}^{3d}} \frac{1\!\!1_{W_{n\ominus 2R}}(u) 1\!\!1_{W_{n\ominus 2R}}(v)}{||v-u||^{d-1}||v-w||^{d-1}} \widetilde{J}(||v-u||,||v-w||)\widetilde{L}(u,v,w,\mathbf{X}) \\
&\quad \times \Phi_1(||v-u||,||w-v||) K_1\left(\frac{||v-u||-r}{b_n}\right) K_1\left(\frac{||v-w||-r}{b_n}\right) \mathrm{d}u\mathrm{d}v\mathrm{d}w \\
&= \frac{4}{|W_{n\ominus 2R}|\sigma_d^2} \int_{-r/b_n}^{\infty} \int_{-r/b_n}^{\infty} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \frac{|W_{n\ominus 2R} \cap (W_{n\ominus 2R} - (b_n\rho + r)z)|}{|W_{n\ominus 2R}|} \\
&\quad \times \widetilde{F}(o,(b_n\rho+r)z,(b_n\rho'+r)z')\Phi_1(b_n\rho+r,b_n\rho'+r)K_1(\rho)K_1(\rho')\mathrm{d}\rho\mathrm{d}\rho'\mathrm{d}\sigma_d(z)\mathrm{d}\sigma_d(z').
\end{aligned}$$

The asymptotic behavior of the leading term A_2 is obtained by applying the dominated convergence theorem. When multiplied by $b_n|W_{n\ominus 2R}|$, we get $\lim_{n\to\infty} b_n|W_{n\ominus 2R}|A_2 = 0$.

Next we introduce the finite range property (4) and reasoning analogous with the foregoing on $\lambda(u,v,w,y,\emptyset)$, which ensures that $\lambda(u,v,w,y,\emptyset)$ is a function that depends only variables $||v-u||, ||y-w||, ||w-u||, ||w-v||$), denoted by $\Phi_2(||v-u||,||y-w||,||w-u||,||w-v||)$. We find that

$$\begin{aligned}
A_3 &= \frac{1}{b_n^2 |W_{n\ominus 2R}|^2 \sigma_d^2} \mathrm{E} \int_{\mathbb{R}^4} \frac{1\!\!1_{W_{n\ominus 2R}}(u) 1\!\!1_{W_{n\ominus 2R}}(w)}{||v-u||^{d-1}||w-y||^{d-1}} \widetilde{J}(||v-u||,||w-y||)\widetilde{L}(u,v,w,y,\mathbf{X}) \\
&\quad \times K_1\left(\frac{||v-u||-r}{b_n}\right) K_1\left(\frac{||w-y||-r}{b_n}\right) \lambda(u,v,w,y,\mathbf{X}) \mathrm{d}u\mathrm{d}v\mathrm{d}w\mathrm{d}y \\
&= \frac{1}{|W_{n\ominus 2R}|\sigma_d^2} \int_{\mathbb{R}^d} \int_{-r/b_n}^{\infty} \int_{-r/b_n}^{\infty} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \frac{|W_{n\ominus 2R} \cap (W_{n\ominus 2R} - w)|}{|W_{n\ominus 2R}|} K_1(\rho)K_1(\rho') \\
&\quad \times \Phi_2^{\star}(b_n\rho+r,b_n\rho'+r,||w||,||(b_n\rho+r)z-w||)\mathrm{d}w\mathrm{d}\rho\mathrm{d}\rho'\mathrm{d}\sigma_d(z)\mathrm{d}\sigma_d(z'),
\end{aligned}$$

where

$$\Phi_2^\star(b_n\rho+r,b_n\rho'+r,||w||,||(b_n\rho+r)z-w||)=$$
$$\widetilde{J}(b_n\rho+r,b_n\rho'+r)\widetilde{F}(o,(b_n\rho+r)z,(b_n\rho'+r)z')\Phi_2(b_n\rho+r,b_n\rho'+r,||w||,||(b_n\rho+r)z-w||).$$

By dominated convergence theorem, we get $\lim_{n\to\infty} b_n|W_{n\ominus 2R}|A_3=0$.

For asymptotic behavior of the leading term A_4, it then suffices to repeat the arguments developed previously to conclude the following result.

$$\begin{aligned}
A_4 &= \frac{\beta^{\star 4}}{b_n^2|W_{n\ominus 2R}|^2\sigma_d^2} \\
&\int_{\mathbb{R}^{4d}} 1\!\!1_{W_{n\ominus 2R}}(u)\,1\!\!1_{W_{n\ominus 2R}}(w)\frac{\widetilde{J}(||v-u||,||w-y||)}{||v-u||^{d-1}||w-y||^{d-1}}\,\mathrm{E}[\widetilde{L}(u,v,\mathbf{X})]\,\mathrm{E}[\widetilde{L}(w,y,\mathbf{X})] \\
&\times\Phi(||v-u||)\Phi(||y-w||)K_1\left(\frac{||v-u||-r}{b_n}\right)K_1\left(\frac{||w-y||-r}{b_n}\right)\mathrm{d}u\mathrm{d}v\mathrm{d}w\mathrm{d}y \\
&= \frac{\beta^4}{b_n^2|W_{n\ominus 2R}|\sigma_d^2} \\
&\int_{\mathbb{R}^{3d}} \frac{|W_{n\ominus 2R}\cap(W_{n\ominus 2R}-w)|}{|W_{n\ominus 2R}|||v-u||^{d-1}||w-y||^{d-1}}\widetilde{J}(||v||,||w-y||)\,\mathrm{E}[\widetilde{L}(o,v,\mathbf{X})]\,\mathrm{E}[\widetilde{L}(w,y,\mathbf{X})] \\
&\times\Phi(||v||)\Phi(||y-w||)K_1\left(\frac{||v||-r}{b_n}\right)K_1\left(\frac{||w-y||-r}{b_n}\right)\mathrm{d}v\mathrm{d}w\mathrm{d}y \\
&= \frac{\beta^4}{|W_{n\ominus 2R}|\sigma_d^2} \\
&\int_{\mathbb{R}^d}\int_{-r/b_n}^{\infty}\int_{-r/b_n}^{\infty}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\frac{|W_{n\ominus 2R}\cap(W_{n\ominus 2R}-w)|}{|W_{n\ominus 2R}|}\widetilde{J}(b_n\rho+r,b_n\rho'+r)K_1(\rho)K_1(\rho') \\
&\times\widetilde{F}(o,(b_n\rho'+r)z')\widetilde{F}(o,(b_n\rho+r)z)\Phi(b_n\rho+r)\Phi(b_n\rho'+r)\mathrm{d}w\mathrm{d}\rho\mathrm{d}\rho'\mathrm{d}\sigma_d(z)\mathrm{d}\sigma_d(z').
\end{aligned}$$

Then by dominated convergence theorem, we get $\lim_{n\to\infty} b_n|W_{n\ominus 2R}|A_4=0$. Therefore, we have finished the proof of Theorem 2. □

Proof of Proposition 2 and Proposition 3. The compact set $[r_1,r_2]$ is covered by the intervals $C_i=[s_{i-1},s_i]$, where $s_i=r_1+i(r_2-r_1)/N, i=1,\dots,N$. Choosing N as the largest integer satisfying $N\le c/l_n$ and $l_n=r_nb_n^2$. We apply a triangle inequality decomposition allows for

$$\sup_{s_{i-1}\le r\le s_i}\left|\widehat{R}_n(r)-\mathrm{E}\widehat{R}_n(r)\right| \le \sup_{s_{i-1}\le r,\rho\le s_i}\left|\widehat{R}_n(r)-\widehat{R}_n(\rho)\right| + \sup_{s_{i-1}\le r,\rho\le s_i}\left|\mathrm{E}\widehat{R}_n(r)-\mathrm{E}\widehat{R}_n(\rho)\right| + \sup_{s_{i-1}\le \rho\le s_i}\left|\widehat{R}_n(\rho)-\mathrm{E}\widehat{R}_n(\rho)\right|.$$

By Lipschitz condition (Condition $\mathscr{L}p$), we derive that there exists constant $\eta>0$ such that n sufficiently large

$$\begin{aligned}
&\sup_{s_{i-1}\le r,\rho\le s_i}\left|\widehat{R}_n(r)-\widehat{R}_n(\rho)\right| \\
&\le \frac{1}{b_n^2}\eta|r-\rho|\frac{1}{\sigma_d|W_{n\ominus 2R}|}\sum_{\substack{u,v\in\mathbf{X}\\ ||v-u||\le R}}^{\neq}\frac{1\!\!1_{W_{n\ominus 2R}}(u)}{||v-u||^{d-1}}\widetilde{h}(u,\mathbf{X}\setminus\{u,v\})\widetilde{h}(v,\mathbf{X}\setminus\{u,v\}). \\
&\le \eta r_n\frac{1}{\sigma_d|W_{n\ominus 2R}|}\sum_{\substack{u,v\in\mathbf{X}\\ ||v-u||\le R}}^{\neq}\frac{1\!\!1_{W_{n\ominus 2R}}(u)}{||v-u||^{d-1}}\widetilde{h}(u,\mathbf{X}\setminus\{u,v\})\widetilde{h}(v,\mathbf{X}\setminus\{u,v\}).
\end{aligned}$$

Follows from the last inequalities and the Nguyen and Zessin ergodic theorem [19], we get

$$\sup_{s_{i-1}\le r,\rho\le s_i}\left|\widehat{R}_n(r)-\widehat{R}_n(\rho)\right| = \mathscr{O}_{a.s.}(r_n) \quad \text{as} \quad n\to\infty.$$

As well, we obtain

$$\sup_{s_{i-1}\le r,\rho\le s_i}\left|\mathrm{E}\widehat{R}_n(r)-\mathrm{E}\widehat{R}_n(\rho)\right| = \mathscr{O}_{a.s.}(r_n) \quad \text{as} \quad n\to\infty.$$

We now concentrating on the stochastic part.

Lemma 1. *Assume that either* (10) *holds for some* $0<q<2$ *such that* $\bar{R}_0\in\mathbb{L}_{\psi_{\theta(q)}}$ *and* $r_n=(\log n)^{1/q}/b_n(\sqrt{n})^d$. *Then*

$$\sup_{s_{i-1}\le \rho\le s_i}\left|\widehat{R}_n(\rho)-\mathrm{E}\widehat{R}_n(\rho)\right| = \mathscr{O}_{a.s.}(r_n) \quad as \quad n\to\infty.$$

Proof. We consider the exponential Young function define for any $t \in \mathbb{R}^+$ by $\psi_q(t) = \exp((t+\xi_q)^q) - \exp(\xi_q^q)$ where $\xi_q = ((1-q)/q)^{1/q} 1\!\!1\{0<q<1\}$. Let $\varepsilon > 0$ and $r \in [r_1, r_2]$ be fixed

$$\begin{aligned}\mathbb{P}\left(|\widehat{R}_n(r) - \mathrm{E}\widehat{R}_n(r)| > \varepsilon r_n\right) &= \mathbb{P}\left(|S_n| > \varepsilon r_n b_n n^d\right) \\ &\leq \exp\left[-\left(\frac{\varepsilon\, r_n b_n n^d}{||S_n||_{\psi_{\theta(q)}}} + \xi_q\right)^q\right] \mathrm{E}\exp\left[\left(\frac{|S_n|}{||S_n||_{\psi_{\theta(q)}}} + \xi_q\right)^q\right].\end{aligned}$$

Therefore, we assume that there exists a real $0 < q < 2$, such that $\bar{R}_0 \in \mathbb{L}_{\psi_{\theta(q)}}$ and using Kahane-Khintchine inequalities (cf. [7], Theorem 1), we have

$$\begin{aligned}\mathbb{P}\left(|\widehat{R}_n(r) - \mathrm{E}\widehat{R}_n(r)| > \varepsilon r_n\right) &= \mathbb{P}\left(|S_n| > \varepsilon r_n b_n n^d\right) \\ &\leq (1+e^{\xi_q^q}) \exp\left[-\left(\frac{\varepsilon\, r_n b_n n^d}{M(\sum_{i\in\Gamma_n} b_{i,q}(\bar{R}))^{1/2}} + \xi_q\right)^q\right]\end{aligned}$$

denote

$$b_{i,q}(\bar{R}) = \left\|\bar{R}_0\right\|^2_{\psi_{\theta(q)}} + \sum_{k\in V_0^1} \left\|\sqrt{\left|\bar{R}_k E_{|k|}(\bar{R}_0)\right|}\right\|^2_{\psi_{\theta(q)}}.$$

We derive that if condition (10) holds, then there exists constant $C > 0$ and so if $r_n = (\log n)^{1/q}/b_n(\sqrt{n})^d$

$$\sup_{r_1\leq r\leq r_2} \mathbb{P}(|\widehat{R}_n(r) - \mathrm{E}\widehat{R}_n(r)| > \varepsilon r_n) \leq (1+e^{\xi_q^q}) \exp\left[-\frac{\varepsilon^q \log n}{C^q}\right].$$

Choosing ε sufficiently large, therefore, it follows with Borel-Cantelli's lemma

$$\mathbb{P}(\limsup_{n\to\infty} \sup_{s_{i-1}\leq\rho\leq s_i} \left|\widehat{R}_n(\rho) - \mathrm{E}\widehat{R}_n(\rho)\right| > \varepsilon r_n) = 0.$$

$\square$

Now, we will accomplish the proof of Proposition 3.

Lemma 2. *Assume* (11) *holds for some* $p > 2$ *such that* $\bar{R}_0 \in \mathbb{L}^p$ *and* $b_n = n^{-q_2}(\log n)^{q_1}$ *for some constants* $q_1, q_2 > 0$. *Let* $a, b \geq 0$ *be fixed and denote* $r_n = n^a(\log n)^b/b_n(\sqrt{n})^d$. *If*

$$a(p+1) - d/2 - q_2 > 1 \quad \textit{and} \quad b(p+1) + q_1 > 1,$$

then

$$\sup_{s_{i-1}\leq\rho\leq s_i} \left|\widehat{R}_n(\rho) - \mathrm{E}\widehat{R}_n(\rho)\right| = \mathcal{O}_{a.s}(r_n) \quad \textit{as} \quad n\to\infty.$$

Proof. Let $p > 2$ be fixed, such that $\bar{R}_0 \in \mathbb{L}^p$ and for any $\varepsilon > 0$,

$$\begin{aligned}
\mathbb{P}(|\widehat{R}_n(r) - \mathrm{E}\widehat{R}_n(r)| > \varepsilon r_n) &= \mathbb{P}\left(|S_n| > \varepsilon r_n b_n n^d\right) \\
&\leq \frac{\varepsilon^{-p}\,\mathrm{E}|S_n|^p}{r_n^p b_n^p n^{pd}} \\
&\leq \frac{\varepsilon^{-p}}{r_n^p b_n^p n^{pd}}\left(2p\sum_{i\in\Gamma_n} c_i(\bar{R})\right)^{p/2}.
\end{aligned}$$

The last inequality follows from a Marcinkiewicz-Zygmund type inequality by [4], where

$$c_i(\bar{R}) = \|\bar{R}_i\|_p^2 + \sum_{k\in V_i^1} \|\bar{R}_k E_{|k-i|}(\bar{R}_i)\|_{\frac{p}{2}}.$$

Under assumption (11) and with the stationarity of $\mathbf{X}$, we derive that there exists $C > 0$ such that

$$\begin{aligned}
\mathbb{P}\left(\sup_{s_{i-1}\leq\rho\leq s_i}\left|\widehat{R}_n(\rho) - \mathrm{E}\widehat{R}_n(\rho)\right| > \varepsilon r_n\right) &\leq N \sup_{r_1\leq r\leq r_2} \mathbb{P}(|\widehat{R}_n(r) - \mathrm{E}\widehat{R}_n(r)| > \varepsilon r_n) \\
&\leq N\frac{\kappa\varepsilon^{-p}}{r_n^p b_n^p (\sqrt{n})^{pd}}.
\end{aligned}$$

As $N \leq c/l_n$ and $l_n = r_n b_n^2$, then for $r_n = n^a(\log n)^b / b_n(\sqrt{n})^d$, it results for n sufficiently large,

$$\begin{aligned}
\mathbb{P}\left(\sup_{s_{i-1}\leq\rho\leq s_i}\left|\widehat{R}_n(\rho) - \mathrm{E}\widehat{R}_n(\rho)\right| > \varepsilon r_n\right) &\leq \frac{\kappa\varepsilon^{-p}}{n^{a(p+1)-d/2}(\log n)^{b(p+1)} b_n} \\
&\leq \frac{\kappa\varepsilon^{-p}}{n^{a(p+1)-d/2-q_2}(\log n)^{b(p+1)+q_1}}.
\end{aligned}$$

For $a(p+1) - d/2 - q_2 > 1$ and $b(p+1) + q_1 > 1$, we get for any $\varepsilon > 0$

$$\sum_{n\geq 1} \mathbb{P}\left(\sup_{s_{i-1}\leq\rho\leq s_i}\left|\widehat{R}_n(\rho) - \mathrm{E}\widehat{R}_n(\rho)\right| > \varepsilon r_n\right) < \infty.$$

□

Considering these arguments the proofs of Proposition 2 and Proposition 3 are completed, it results from a direct application of the theorem of Borel-Cantelli and by Theorem 1:

$$\sup_{r_1\leq r\leq r_2} |\mathrm{E}\widehat{R}_n(r) - R(r)| = \mathscr{O}(b_n^\alpha) \quad \text{as} \quad n\to\infty.$$

□

Proof of Proposition 1. We consider the mean square error of $\widehat{R}_n(r)$,

$$\mathrm{MSE}\left(\widehat{R}_n(r)\right) = \mathrm{Var}\left(\widehat{R}_n(r)\right) + \left(Biais(\widehat{R}_n(r)\right)^2.$$

The estimator $\widehat{R}_n(r)$ is asymptotically unbiased by Theorem 1 and so we have

$$\left(Biais(\widehat{R}_n(r)\right)^2 = \left(\mathrm{E}\widehat{R}_n(r) - \beta^{\star 2} J(r)\exp(-\gamma(r))\right)^2 \longrightarrow 0$$

and by Theorem 2, we have

$$\mathrm{Var}\left(\widehat{R}_n(r)\right) = \mathrm{E}\left(\widehat{R}_n(r) - \mathrm{E}\widehat{R}_n(r)\right)^2 \longrightarrow 0.$$

Hence, $\widehat{R}_n(r)$ is consistent in the quadratic mean and hence consistent estimate of $\beta^{\star 2} J(r)\exp(-\gamma(r))$. $\widehat{J}_n(r)$ and $\widehat{\beta}_n$ are strongly consistent estimates of $\beta^\star J(r)$ and $\beta^\star$, thus the consistency (convergence in probability) of the estimator $\widehat{\Phi}_n(t)$ of $\exp(-\gamma(r))$. Then one gets with the same arguments as before and by Proposition 2 or Proposition 3, we conclude the strong consistency of the estimator $\widehat{\Phi}_n(r)$ of the function $\exp(-\gamma(r))$. $\square$

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